Markov chains model reduction

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Seminar on Stochastic Processes 2016
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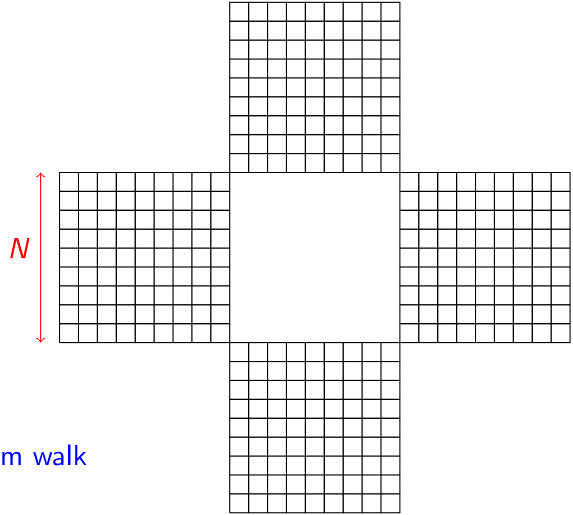
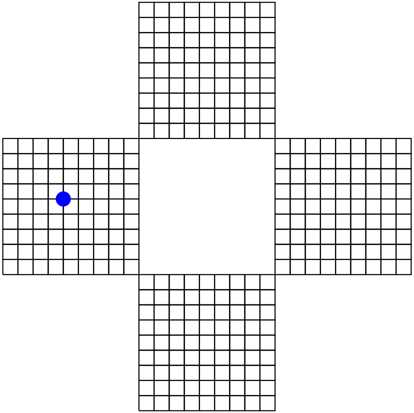
Model Reduction



$\eta^N(t)$ random walk

Model Reduction

- Identification of the slow variables
- Derive their asymptotic behavior



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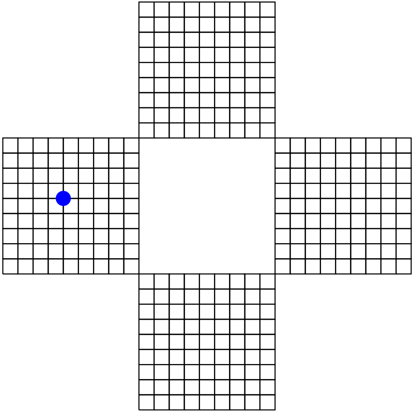
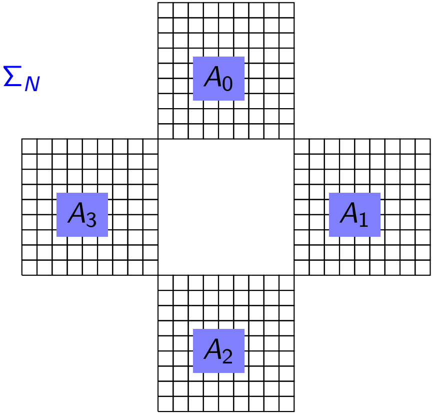


- $d = 2$ Mixing time N^2 Hitting time $N^2 \log N$
- $d \geq 3$ Mixing time N^2 Hitting time N^d

Model Reduction



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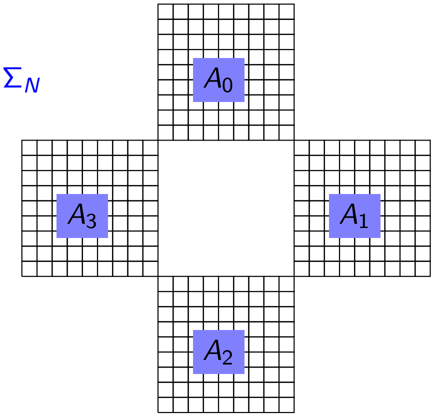
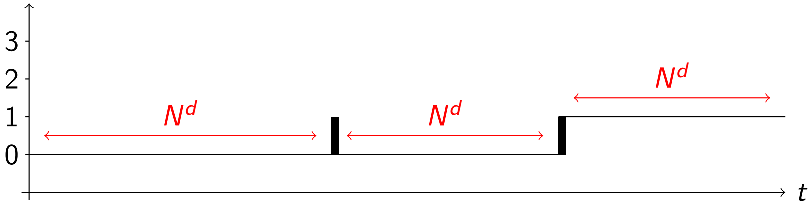


- $\Psi_N : \Sigma_N \to \{0, 1, 2, 3\}$ $\quad \Psi_N(x) = \sum_{k=0}^{3} k\, \mathbf{1}\{x \in A_k\}$
- $X_N(t) = \Psi_N(\eta^N(t))$

Model Reduction

- $X_N(t)$ not Markov   Hidden Markov Chain



Model Reduction

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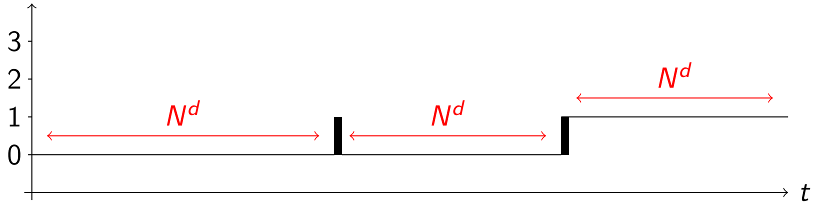
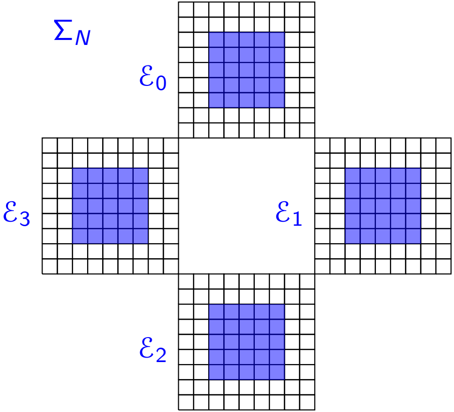


- No convergence $X_N(t)$ in Skorohod topology

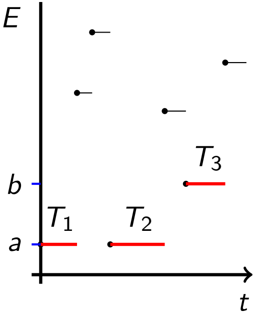
Trace process

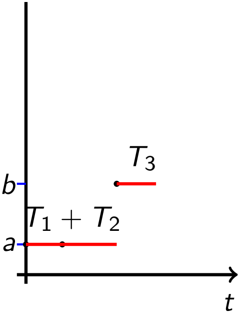


- $\Delta_N = \Sigma_N \setminus [\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3]$

Trace of a Markov chain

- $\eta(t)$ E-valued Markov chain $F \subsetneq E$
- $\eta^F(t)$ trace of $\eta(t)$ on F $F = \{a, b\}$





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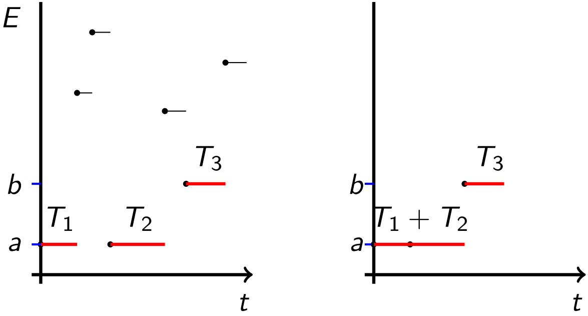
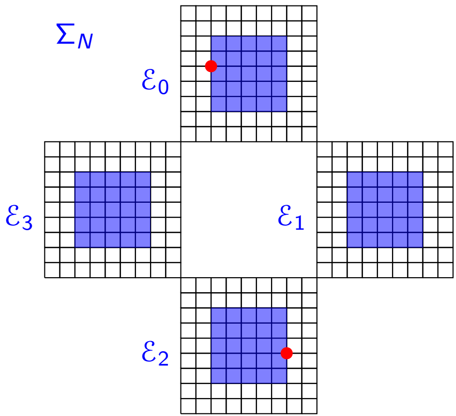


- $\eta^F(t)$ Markov chain $r_F(x, y) = \lambda(x)\, \mathbb{P}_x[H_{F\setminus\{x\}} = H_y]$

Trace process



- $\Delta_N = \Sigma_N \setminus [\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3]$
- $r_F(x,y) = \lambda(x)\, \mathbb{P}_x[H_{F\setminus\{x\}} = H_y]$ long jumps

Problem

- $\eta^{\mathcal{E}}(t)$ trace of $\eta(t)$ on $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$
- $\Psi_N : \mathcal{E} \to \{0, 1, 2, 3\} \quad \Psi_N(x) = \sum_{k=0}^{3} k\, \mathbf{1}\{x \in \mathcal{E}_k\}$
- $X^N(t) = \Psi_N(\eta^{\mathcal{E}}(t))$ Hidden M. C.

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- $X^N(\theta_N t) \to X(t)$
- $\max_{x \in \mathcal{E}} \mathbb{E}_x \Big[\int_0^t \mathbf{1}\{\eta(\theta_N s) \notin \mathcal{E}\}\, ds \Big] \to 0$

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- $\eta(\theta_N t) \to X(t)$ soft topology

Martingale approach

Beltrán, L. (2010-15)

- $X^N(t) = \Psi_N(\eta^{\mathcal{E}}(t\theta_N)) \to X(t)$

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- $F(X_t) - F(X_0) - \int_0^t (\mathcal{L}F)(X_s)\, ds$ martingale $F : \{0, \dots, 3\} \to \mathbb{R}$

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- $F(X_t) - F(X_0) - \int_0^t \sum_{k=0}^{3} r(X_s, k)\, [F(k) - F(X_s)]\, ds$ martingale
- $H : \mathcal{E} \to \mathbb{R}$
- $H(\eta^{\mathcal{E}}(\theta_N t)) - H(\eta^{\mathcal{E}}(0)) - \int_0^{t\theta_N} [L_{\mathcal{E}} H]\, (\eta^{\mathcal{E}}(s))\, ds$
- $[L_{\mathcal{E}} H](\eta) = \sum_{\xi \in \mathcal{E}} R^{\mathcal{E}}(\eta, \xi)\, \{H(\xi) - H(\eta)\}$

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- $[L_{\mathcal{E}} H](\eta) = \sum_{\xi \in \mathcal{E}} R^{\mathcal{E}}(\eta, \xi)\, \{H(\xi) - H(\eta)\}$
- $H = F \circ \Psi_N$
- $F(X^N(t)) - F(X^N(0)) - \theta_N \int_0^t [L_{\mathcal{E}}(F \circ \Psi)](\eta^{\mathcal{E}}(\theta_N s))\, ds$

Closing the equation

- $\theta_N \int_0^t [L_{\mathcal{E}}(F \circ \Psi)](\eta^{\mathcal{E}}(\theta_N s))\, ds \to \int_0^t \sum_{k=0}^{3} r(X_s, k)\, [F(k) - F(X_s)]\, ds$

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$$
\begin{aligned}
[L_{\mathcal{E}}(F \circ \Psi)](\eta) &= \sum_{\xi \in \mathcal{E}} R^{\mathcal{E}}(\eta, \xi)\, \{(F \circ \Psi)(\xi) - (F \circ \Psi)(\eta)\} \\
&= \sum_{k=0}^{3} [F(k) - F(X^N)] \sum_{\xi \in \mathcal{E}_k} R^{\mathcal{E}}(\eta, \xi) \\
&= \sum_{k=0}^{3} [F(k) - F(X^N)]\, R^{\mathcal{E}}(\eta, \mathcal{E}_k)
\end{aligned}
$$

Closing the equation

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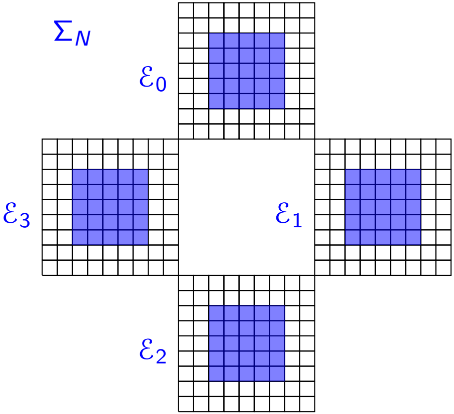
\end{aligned}$$

- $$\int_0^t \theta_N\, R^{\mathcal{E}}(\eta^{\mathcal{E}}(\theta_N s), \mathcal{E}_k)\, ds \sim \int_0^t r(X_s^N, k)\, ds$$

Local ergodicity

- $\int_0^t \theta_N \, R^{\mathcal{E}}(\eta^{\mathcal{E}}(\theta_N s), \mathcal{E}_k) \, \mathbf{1}\{X_s^N \neq k\} ds \;\sim\; \int_0^t r(X_s^N, k) \, \mathbf{1}\{X_s^N \neq k\} \, ds$



Local ergodicity

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- $G : \mathcal{E} \to \mathbb{R}$
- $\widehat{G} = E_{\pi_N}[G(\eta) | \mathcal{E}_0, \dots, \mathcal{E}_3]$ $\quad \pi_N$ ss
- $\widehat{G}(\eta) \;=\; \dfrac{1}{\pi_N(\mathcal{E}_j)} \displaystyle\sum_{\eta \in \mathcal{E}_j} \pi_N(\eta) G(\eta) \qquad \eta \in \mathcal{E}_j$
- $\widehat{G}(\eta) = \widehat{G}(\Psi_N(\eta))$

$$\int_0^t \Big\{ G(\eta^{\mathcal{E}}(\theta_N s)) - \widehat{G}(\eta^{\mathcal{E}}(\theta_N s)) \Big\} \, ds \;\longrightarrow\; 0 \qquad \textbf{(C1)}$$

Consequence

- $\displaystyle\int_0^t \theta_N\, R^{\mathcal{E}}(\eta^{\mathcal{E}}(\theta_N s), \mathcal{E}_k)\, \mathbf{1}\{X_s^N \neq k\} ds \ \sim\ \int_0^t r(X_s^N, k)\, \mathbf{1}\{X_s^N \neq k\}\, ds$
- $\displaystyle\int_0^t \theta_N\, R^{\mathcal{E}}(\eta^{\mathcal{E}}(\theta_N s), \mathcal{E}_k)\, \mathbf{1}\{X_s^N = j\} ds \ \sim\ \int_0^t r(j, k)\, \mathbf{1}\{X_s^N = j\}\, ds$

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$$\frac{\theta_N}{\pi_N(\mathcal{E}_j)} \sum_{\eta \in \mathcal{E}_j} \pi_N(\eta) R^{\mathcal{E}}(\eta, \mathcal{E}_k) \ =: \ \theta_N \, r_N(j,k)$$

$$\theta_N \, \frac{1}{\pi_N(\mathcal{E}_j)} \sum_{\eta \in \mathcal{E}_j} \pi_N(\eta) R^{\mathcal{E}}(\eta, \mathcal{E}_k) \to r(j,k) \qquad \textbf{(C2)}$$

- $\displaystyle\theta_N \int_0^t [L_{\mathcal{E}}(F \circ \Psi)](\eta^{\mathcal{E}}(\theta_N s))\, ds \ \to \ \int_0^t (LF)(X_s)\, ds$

Summary

- $\widehat{G}(\eta) \;=\; E_{\pi_N}[G(\eta)|\mathcal{E}_0,\dots,\mathcal{E}_3]$

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$$\max_{x \in \mathcal{E}} \mathbb{E}_x \Big[\int_0^t \mathbf{1}\{\eta(\theta_N s) \notin \mathcal{E}\}\, ds \Big] \to 0 \qquad \textbf{(C3)}$$

- $\Psi_N(\eta(t\theta_N)) \to X_t$ soft topology

Applications

- Random walks
  - Random walks random traps
  - Random walks in potential field
- Spin models
  - Kawasaki dynamics in $d = 2$
  - Blume-Capel dynamics
  - Mean field Potts models
- Interacting particle systems
  - Condensing zero-range processes
  - ABC model
  - Inclusion-exclusion dynamics
- Random polymers
  - In interaction with substrates
  - repulsive wall
- Reversible and non-reversible dynamics
- Logarithmic barriers

Tools

Processes which visit points:

- Potential theory
  - (C1)–(C3) formulated in terms of capacities

General:

- Ergodic properties
  - Spectral gap

Condensing zero-range processes

Model:

- $\mathbb{T}_L = \{1, \dots, L\}$ periodic boundary conditions
- State space $\mathbb{N}^{\mathbb{T}_L}$
- configurations $\eta = \{\eta_x : x \in \mathbb{T}_L\}$

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Dynamics:

- $g : \mathbb{N} \to \mathbb{R}_+$ $g(0) = 0$ $g(k) > 0$ $0 < p \leq 1$
- $x \to x + 1$ at rate $pg(\eta_x)$ $x \to x - 1$ at rate $(1 - p)g(\eta_x)$
- $g(k) = k$ independent random walks
- $g(k) = \mathbf{1}\{k \geq 1\}$ queues and servers

Condensation

Evans (2005), Armendariz, Beltran, Chleboun, Ferrari, Godrèche, Grosskinsky, Loulakis, Schuetz, Sisko, Spohn

- $g(1) = 1 \quad g(k) = \Big(\dfrac{k}{k-1}\Big)^{\alpha} \quad k \geq 2 \quad \alpha > 0$
- N number of particles
- $E_{L,N} = \{\eta \in \mathbb{N}^{\mathbb{T}_L} : \sum_{x\in\mathbb{T}_L} \eta_x = N\}$
- $\{\eta(t) : t \geq 0\}$ irreducible $\quad \mu_{L,N}$

Phase transition

- $\alpha > 2$ T removes the site with largest number of particles
- $\{N_L : L \geq 1\} \quad N_L/L \to \rho > \rho^* \quad \mu_{L,N} T^{-1} \sim \nu_{\rho^*} \quad L(\rho - \rho^*)$

- $\alpha > 1 \quad L$ fixed
- $1 \ll \ell_N \ll N \quad \lim_{N\to\infty} \mu_{L,N}\{\max_{1\leq x\leq L} \eta_x \geq N - \ell_N\} = 1$

Model Reduction, Coarse graining

- Fix $1 \ll \ell_N \ll N$
- $\mathcal{E}_N^x = \{\eta : \eta_x \geq N - \ell_N\} \quad 1 \leq x \leq L$
- $\mathcal{E}_N = \bigcup_{x=1}^{L} \mathcal{E}_N^x \quad E_{L,N} \;=\; \mathcal{E}_N \,\cup\, \Delta_N$

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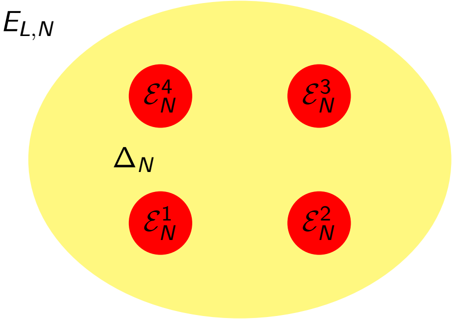

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- $X_N(t) = \Psi_N(\eta^{\mathcal{E}_N}(N^{1+\alpha} t)) \to X(t)$

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- $\theta_N = N^{1+\alpha}$
- Reversible: $r(x, y) = C(\alpha)\, L\, \mathrm{cap}(x, y)$
- Non-reversible $\quad p = 1$