Washington Experimental Mathematics Lab

Brownian Bridges

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Introduction to the Theory of Brownian Bridges

Definition An m-dimensional Wiener (or Brownian Motion) process with mean μ and variance σ^2 is a stochastic process $(W_t)_{t\geq 0}$ with state space $\mathbb{R}^m$ satisfying:

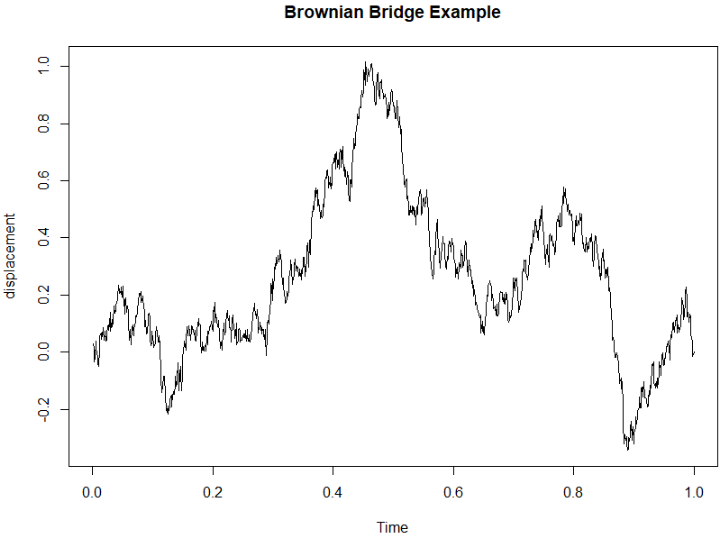
1. $\forall \omega \in \Omega, W_0(\omega) = 0$;
2. $\forall 0 \leq t_1 < t_2 < t_3 < \ldots < t_n$: $W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent
3. $\forall 0 \leq s < t, W_t - W_s \sim \mathcal{N}(\mu(t-s), \sigma^2 (t-s))$;
4. $\forall \omega \in \Omega, W^\omega : [0, \infty) \to \mathbb{R}^m$ is continuous

Remark Let $(W_t)_{t\geq 0}$ be an m-dimensional Wiener process with mean μ, variance σ^2, and let $T > 0$. Define $(B_t)_{t\in[0,T]}$ by

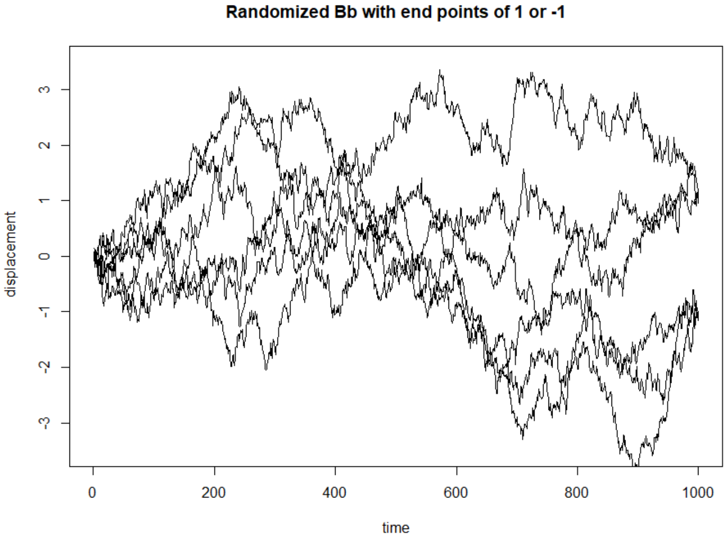
$$B_t \equiv W_t - \frac{t}{T} W_T.$$

$(B_t)_{t\in[0,T]}$ is called a Brownian bridge on $[0, T]$ with parameters μ, σ^2.

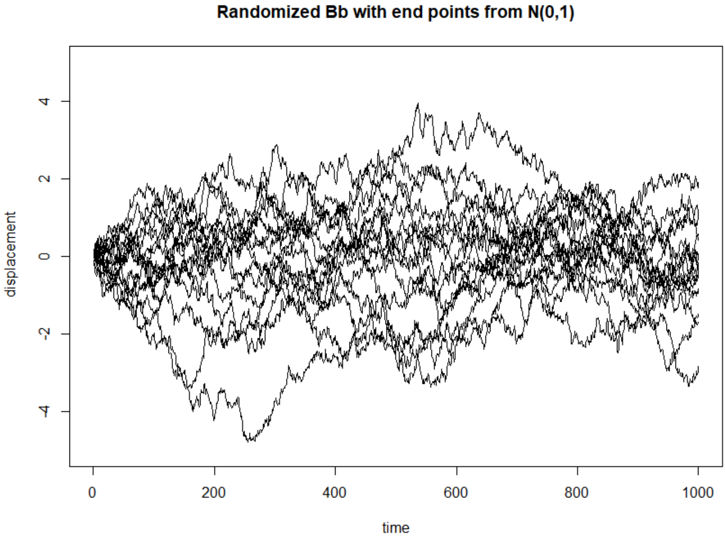
Brownian Motion



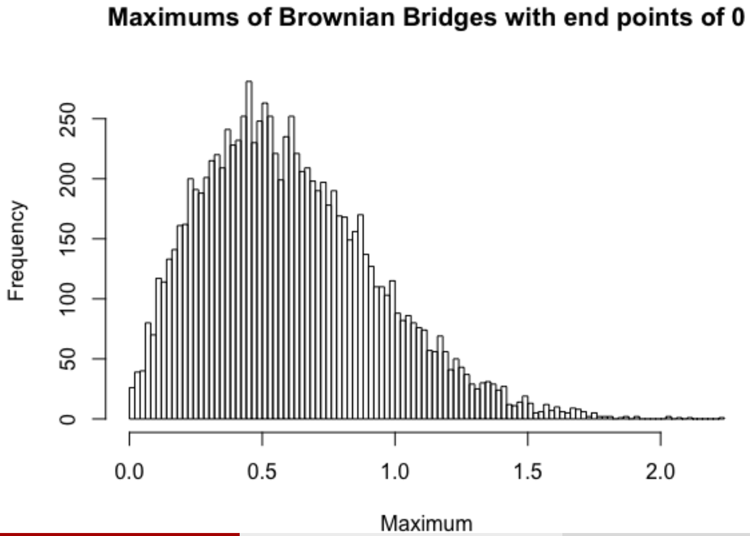
1D Randomized End Points



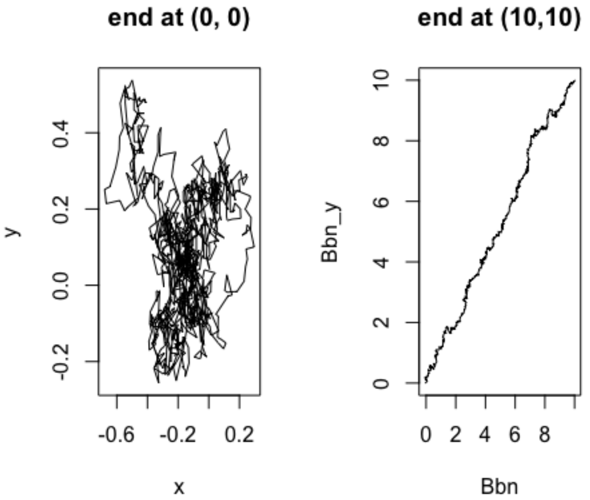
1D End Points From Random Normal



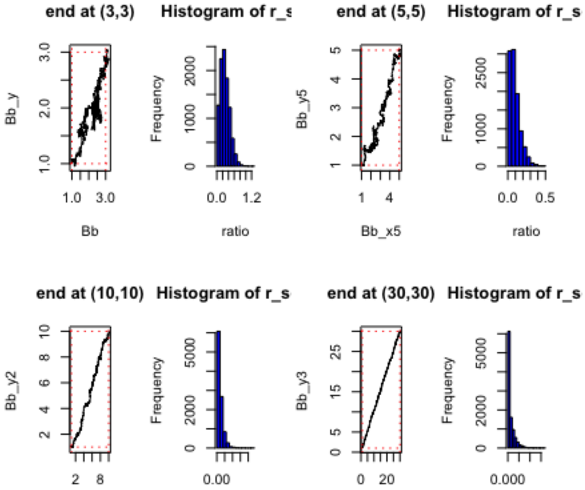
1D Distribution of Maximum Values



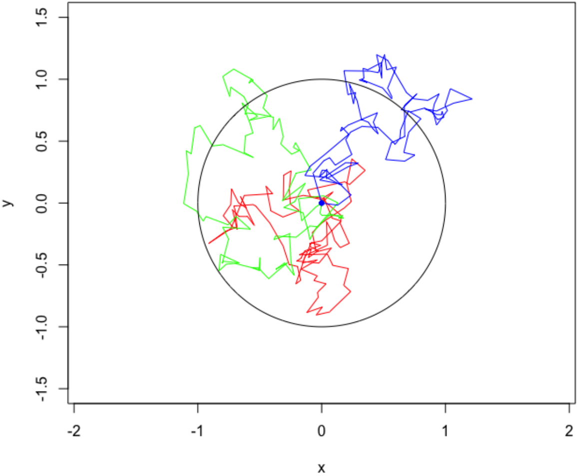
2D Brownian Bridges Fixed Start End Point



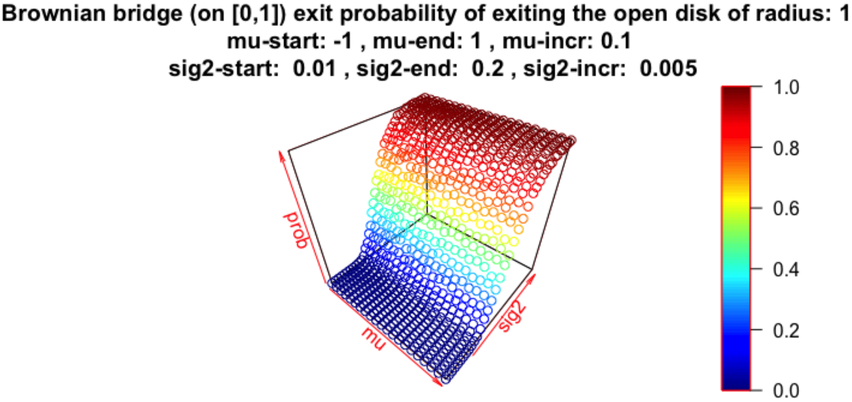
2D Brownian Bridges different End Point AND Probability of exist



2D Brownian Bridges Example



2D Brownian Bridges Exit Probabilities from the Open Unit Disk



2D Brownian Bridges First Exit Times Distributions Evolution

link

https://goo.gl/XV5kMq

Sources

- https://www.math.ucdavis.edu/ hunter/m280_09/ch5.pdf
- https://en.wikipedia.org/wiki/Wiener_process
- https://en.wikipedia.org/wiki/Brownian_bridge
- http://www.columbia.edu/ ks20/FE-Notes/4700-07-Notes-B